$\dot{T}/T = S$. From the boundary condition, it appears that $\lambda_n = n\pi$ for $n = \pm 1, \pm 2, \ldots$. Therefore, the characteristic equation can be reduced to

$$S^2 + S(\nu_H + \nu)\frac{\lambda_n^2}{\ell^2} + V_n^2\frac{\lambda_n^2}{\ell^2} + \nu\,\nu_H\frac{\lambda_n^4}{\ell^4} = 0$$

for the transverse mode with the roots

$$S_n\|_{1,2} = \tfrac{1}{2}\left[-(\nu_H + \nu)\frac{\lambda_n^2}{\ell^2} \pm\sqrt{(\nu_H - \nu)^2\frac{\lambda_n^4}{\ell^4} - 4V_x^2\frac{\lambda_n^2}{\ell^2}}\right]$$

This expression for S_n indicates that $\sup[Re\,\sigma(A)] < 0$, which is the necessary and sufficient condition for the equilibrium solution of system equations to be exponentially stable, i.e., an equivalence to uniform asymptotic stability for the preceding linear system.

The point spectrum approach, applied to the longitudinal mode, results in the following spectrum of the set of state equations:

$$\left\{K\left(\frac{1}{\rho_0} + 4/3\frac{\nu}{p_0}S\right)\lambda_n^4 + \left[S^2\frac{K}{p_0} + 4/3\frac{\nu S^2}{T_0(\gamma-1)} + SC_p\right]\lambda_n^2 + \frac{S^3}{T_0(\gamma-1)}\right\}\left\{(\nu_H\lambda_n^2 + S)(\nu\lambda_n^2 + S) + V_x^2\lambda_n^2\right\} + \left\{\lambda_n^2V_y^2(S + \nu\lambda_n^2)\right\}\left\{\frac{S^2}{T_0(\gamma-1)} + \frac{S\,K\lambda_n^2}{p_0}\right\} = 0$$

where p_0 is the pressure related to ρ_0 and T_0, and γ equal C_p/C_v.

This characteristic equation does not have a closed-form solution. Therefore, in general, the spectrum approach is very complex and requires a very cumbersome symbolic manipulation in order to give stability results except for very simplified or special cases. Also, in general, λ can be both positive and negative numbers, which results in two sets of characteristic equations with positive and negative coefficients. The outlined Lyapunov approach however, would result in stability solutions for the system without any resort to the system parameters and the form of wave number.

Conclusion

A stability analysis of a magnetoplasmadynamic system was presented. This technique is based on the Lyapunov theorem extended to cover distributed parameter systems. The results of the stability analysis were supported by those derived from a spectral analysis point of view. The procedure for construction of the Lyapunov functional and derivation of its derivative was presented. It was shown that although the spectral technique can be applied to stability analysis of a special class of systems, the presented method can be applied to any form of distributed parameter systems.

Acknowledgment

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Calculation of Structural Dynamic Forces and Stresses Using Mode Acceleration

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Introduction

ONE challenge in structural dynamics is to accurately calculate displacement-related quantities such as element forces and stresses during a transient analysis while using a small number of modal degrees of freedom. A method for increasing the accuracy of the calculation is the use of mode acceleration rather than mode displacement data recovery.[1-3] This approach combines an exact static representation of the structure with a dynamic correction factor based on the modal accelerations. The standard mode acceleration formulation has often been interpreted to suggest that the reason for improved convergence (improved accuracy with a smaller number of modes) is that the dynamic correction factor is divided by the modal frequencies squared.[1,2] Here, we present an alternate formulation that indicates clearly that the only difference between mode acceleration and mode displacement data recovery is the addition of a static correction term. Reference 3 provides a correct interpretation of the method, though the alternate formulation presented here illustrates this more clearly. This alternate formulation also shows clearly that the use of the mode acceleration method is especially important when large input forces are present and, conversely, that mode acceleration and mode displacement data recovery are identical when input forces are not present (i.e., during free decay). We discuss some advantages in numerical implementation associated with the alternate formulation, and provide a simple example.

Derivation and Discussion

The standard mode acceleration formulation for a structure with proportional damping is[1]

$$x = \Psi f - 2\Phi Z\Omega^{-1}\dot{q} - \Phi\Omega^{-2}\ddot{q} \tag{1}$$

where

x = vector of displacement-related quantities (e.g., element forces or stresses)
$\Psi - \Psi_{ij}$ = static response of x_i due to a unit force f_j
f = vector of input forces to the structure
$\Phi - \Phi_{ij}$ = static response of x_i due to unit deflection in the modal degree of freedom q_j
Z = diagonal matrix of modal damping ratios, $Z = \text{diag}\{\zeta_\theta\}$
Ω = diagonal matrix of modal frequencies, $\Omega = \text{diag}\{\omega_i\}$
q = vector of modal displacements

The modal damping term is often neglected for lightly damped structures, but we will incorporate it in this formulation. Now consider the corresponding dynamic equations of motion in modal form:

$$\ddot{q} + 2Z\Omega\dot{q} + \Omega^2 q = \Gamma^T f \tag{2}$$

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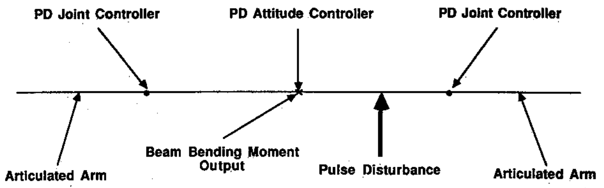

Fig. 1 Example structure.

where Γ are modal shape coefficients at location of input forces.

Equation (2) can be solved for $\ddot{q}$ as follows:

$$\ddot{q} = \Gamma^T f - 2Z\Omega\dot{q} - \Omega^2 q \tag{3}$$

and substituted into Eq. (1), resulting in the following alternate formulation for mode acceleration data recovery:

$$x = (\Psi - \Phi\Omega^{-2}\Gamma^T)f + \Phi q \tag{4}$$

The $(\Phi\Omega^{-2}\Gamma^T)$ term is simply the static response of the structure based on the truncated modal representation, while Ψ is the exact static response. If the $(\Psi - \Phi\Omega^{-2}\Gamma^T)f$ term is neglected, the remaining Φq term results in standard mode displacement data recovery. Equation (4), therefore, is the standard mode displacement formulation with the addition of a static correction term. Note that Eq. (4) provides identical results to the standard mode acceleration formulation of Eq. (1), but it better illustrates the relationship between mode acceleration and mode displacement data recovery methods. In particular, the difference is not that one uses modal displacements while the other uses modal accelerations. In fact, the mode acceleration method is reformulated here such that modal displacements are used in the calculation. Also, it is not true that the mode acceleration method converges more rapidly because of ω^2 terms in the denominator,[1,2] since modal accelerations are related to modal displacements by an equation that has ω^2 terms in the numerator. The difference between mode acceleration and mode displacement data recovery is simply a static correction term. The alternate formulation also illustrates that the use of mode acceleration rather than mode displacement data recovery is especially important when input forces are large. Conversely, mode acceleration and mode displacement will provide identical answers during a free decay.

The formulation in Eq. (4) may also provide some advantages in practical implementation. The first is that the issue of whether or not modal damping terms are negligible is eliminated, since these terms now drop out of the equations. The second is that the size of the elements in the $(\Psi - \Phi\Omega^{-2}\Gamma^T)$ matrix multiplied by the peak input forces provide an exact measure of the peak difference between mode acceleration and mode displacement data recovery. If these terms are small, the selected modes provide an accurate representation of output quantities, whereas if these terms are large, it is very important to add the static correction term. A final advantage is that the alternate formulation can be incorporated into a state-space model, since the output is a function of the input and state vector only, whereas in the traditional formulation it is a function of the input and the derivative of the state vector. When using a state-space formulation, the static correction terms show up as direct feedthrough terms in the D matrix.

Example

Consider the simple example illustrated in Fig. 1. This is a long beam with two articulated arms constrained to move in a two-dimensional plane. Proportional plus derivative (PD) controllers control the relative rotation at each joint and the overall attitude of the structure. Mass and flexibility properties are chosen to represent a large flexible space structure such as the international space station. A damping ratio of 0.5% is included in the modal degrees of freedom. The structure is excited by a square wave pulse force of 100 lb, on for 1 s, off for 1 s. A simulation is carried out during a 10-s transient in order to determine the bending moment experienced in the center of the beam.

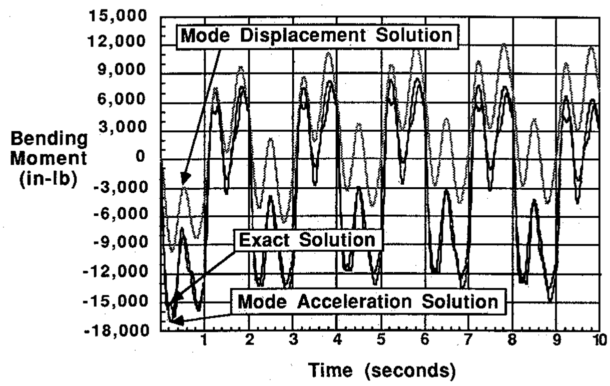

Fig. 2 Comparison of solution methods.

The structure is modeled with finite elements resulting in 29 degrees of freedom including five rigid-body modes. An "exact" solution can, therefore, be calculated by using all 29 degrees of freedom. In this case the static correction matrix $(\Psi - \Phi\Omega^{-2}\Gamma^T)$ reduces to zero, and mode acceleration and mode displacement results are identical. The purpose of this paper, however, is to consider the situation where a truncated mode set is used, therefore, consider a situation where five rigid-body modes and five flexible modes are used in the simulation. This model has four inputs—three control moments and the pulse disturbance—and we are considering one output. After eliminating rigid-body modes from the calculation, the static correction matrix in this case is

$$(\Psi - \Phi\Omega^{-2}\Gamma^T) = [-0.46242, -0.16499, 0.25105, -31.019]$$

The first three inputs are determined by the closed-loop control law, and so they may not be known before the transient simulation. The fourth entry can be multiplied by the maximum disturbance force of 100 lb, however, to give a rough estimate of the error associated with a mode displacement solution of about 3000 in.-lb. The actual results are illustrated in Fig. 2. Note that the mode acceleration solution follows the "exact" solution very closely, whereas the mode displacement solution underestimates the peak response by about 40%. The error is actually larger than 3000 in.-lb due to the effect of applied control moments. The solutions in this case are different even when the pulse disturbance is off. This is because the control systems are still active during this time.

Either formulation of the mode acceleration method provides identical answers. The advantages of the alternate formulation are that the static correction matrix $(\Psi - \Phi\Omega^{-2}\Gamma^T)$ can be used to estimate the error associated with the mode displacement solution, and that the equations can be incorporated directly into control system software that uses a state-space formulation to describe the system. The effect of modal damping is negligible in this case, and so no advantage is gained by the fact that the alternate formulation directly incorporates this term.

Conclusions

We have reformulated the mode acceleration approach for recovery of displacement-related quantities from a structural dynamic transient analysis. The new formulation shows clearly that the mode acceleration approach is identical to the mode displacement approach with the addition of a static correction term. Using this new formulation eliminates the need to neglect modal damping terms, and also facilitates an understanding of the correction made to the mode displacement calculation when using a mode acceleration approach. A simple example has illustrated these results.

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Design of a Modalized Observer with Eigenvalue Sensitivity Reduction

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Introduction

THE modalized observer, which was first proposed by Andry et al.,[1] is based upon the well-known fact that the separation theorem does not extend to the eigenvectors related to the plant and observer. This nonseparation may be taken into account when using eigenstructure assignment to design the observer gain matrix. In particular, the observer eigenvectors can be chosen to minimize the impact of a known mismatch of initial conditions between the observer and plant. This is quite useful in flight control problems where a gust disturbance from straight and level equilibrium flight causes nonzero initial conditions in sideslip angle and angle of attack.

In Ref. 1, a modalized observer was designed for the lateral dynamics of the L-1011 aircraft linearized at some trim condition. A choice of observer eigenvectors was proposed that results in the attenuation of the state estimation error, which is induced by an initial condition mismatch in sideslip angle. However, in this Note we show that the observer in Ref. 1 exhibits a large sensitivity of the observer eigenvalues to parameter variation and/or uncertainty in the aircraft stability and control derivatives. We propose a new method for designing modalized observers that achieve both small eigenvalue sensitivity and attenuation of the estimation error caused by an initial condition mismatch. This new design approach for modalized observers is based on minimizing a cost function, which depends on the norm of the observer modal matrix, the condition number of this modal matrix, and directional information about the initial condition mismatch. We compare this new modalized observer design method with the approach described in Ref. 1 to show the advantages of the new method.

Design Methodology

Consider the linear time invariant plant described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where x is $(n \times 1)$ y is $(r \times 1)$, and u is $(m \times 1)$. Assume that a full-state feedback matrix K has already been obtained by any one of a number of methods. We complete the design by choosing a full-order observer of the form

$$\dot{z}(t) = (A - LC)z(t) + Ly(t) + Bu(t) \tag{3}$$

where L is the observer gain matrix.

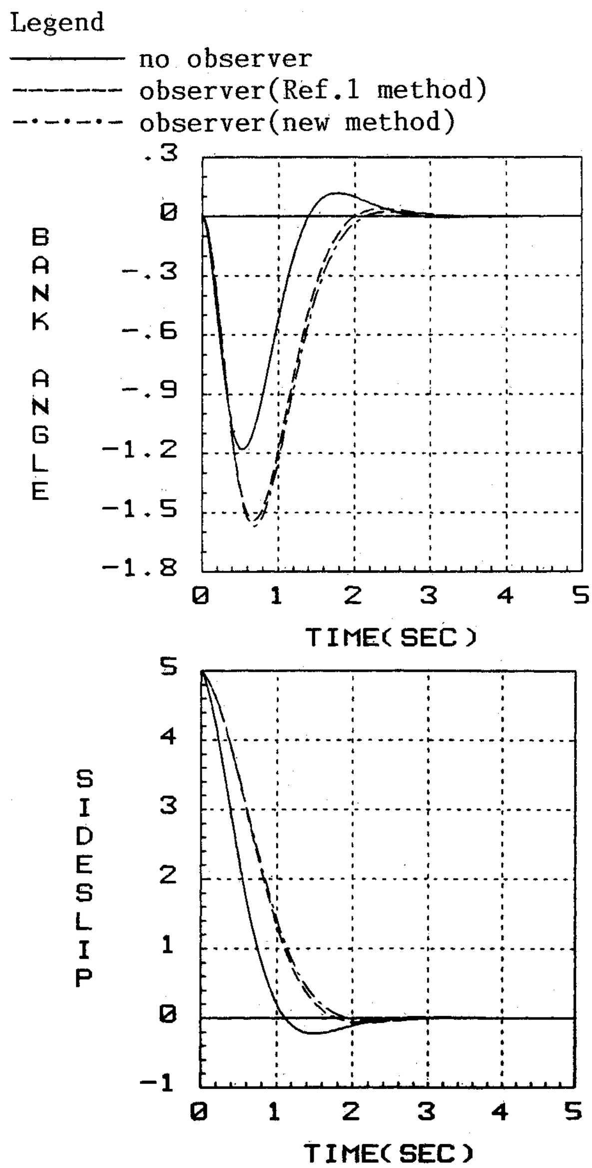

Fig. 1 Aircraft states for three control laws.

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